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Non-commutative geometry of the *h*-deformed quantum plane

S Cho†, J Madore‡ and K S Park§

† Department of Physics, Semyung University, Chechon, Chungbuk 390 - 230, Korea
‡ Laboratoire de Physique Théorique et Hautes Energies‖, Université de Paris-Sud, Bât. 211, F-91405 Orsay, France
§ Department of Mathematics, Keimyung University, Taegu 705 - 701, Korea

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Abstract. The *h*-deformed quantum plane is a counterpart of the *q*-deformed one in the set of quantum planes which are covariant under those quantum deformations of $GL(2)$ which admit a central determinant. We have investigated the non-commutative geometry of the *h*-deformed quantum plane. There is a two-parameter family of torsion-free linear connections, a one-parameter sub-family of which are compatible with a skew-symmetric non-degenerate bilinear map. The skew-symmetric map resembles a symplectic 2-form and induces a metric. It is also shown that the extended *h*-deformed quantum plane is a non-commutative version of the Poincaré half-plane, a surface of constant negative Gaussian curvature.

1. Introduction

Quantum planes are simple examples of quantum spaces and have been studied intensively by many authors in the past years. They arise as deformations of planes on which quantum groups act covariantly. For references to the literature we refer to the recent monographs by Chari and Pressley [5] and by Majid [21]. One of the quantum planes, referred to as the *q*-deformed quantum plane or the Manin plane [22], is defined as the associative algebra generated by two non-commuting elements ('coordinates') x and y such that

$$xy = qyx.$$

The quantum group $GL_q(2)$ is the symmetry group of the *q*-quantum plane. Another quantum plane, called the *h*-deformed quantum plane [7, 23], is defined as the associative algebra generated by two non-commuting elements x and y such that

$$xy - yx = hy^2.$$

The quantum group $GL_h(2)$ is the symmetry group of the *h*-quantum plane. These two quantum planes are the only deformations of the ordinary plane which are covariant under the quantum deformations of $GL(2)$ which admit a central determinant since up to isomorphism $GL_q(2)$ and $GL_h(2)$ are the only two such deformed quantum groups [16]. The *h*-deformation can be seen as a singular contraction of a *q*-deformation [2]. More precisely, a class of similarity transformations of the R-matrices associated to *q*-deformations can be introduced such that the $q \to 1$ limit gives explicit R-matrices for the *h*-deformations [1].

‖ Laboratorie associé au CNRS, URA D0063.

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Although the transformation matrix is itself singular in the limit, the construction is well-defined.

As usual in non-commutative geometry [6, 18] quantum planes have over them many differential calculi $\Omega^*(\mathcal{A})$. The commutation relations in $\Omega^1(\mathcal{A})$ must be consistent with the commutation relations of the algebra but this condition is not enough to uniquely define the calculus. There is, however, a particularly interesting calculus known as the Wess–Zumino calculus [26, 27] which is covariant under the co-action of the q-deformed quantum groups. There is similarly a calculus over the h-deformed quantum plane which is covariant under the co-action of the h-deformed quantum groups [4]. Moreover, general definitions have been proposed recently of a linear connection and a metric within the context of non-commutative geometry in general and for quantum spaces in particular. Using these tools, we shall investigate here the Riemannian geometry of the h-deformed quantum plane. It turns out that the h-deformed quantum plane has more interesting geometrical properties than the q-deformed one.

In section 2 we give a review of the definition of what we call the 'Stehbein' formalism [8, 20] and of a definition of a linear connection [9, 10, 18, 25] which has been used in non-commutative geometry. In section 3, a two-parameter family of torsion-free linear connections is constructed on the h-deformed quantum plane. The existence of a two-parameter family of torsion-free linear connections is shown to be quite general even within the set of two-parameter h-deformed quantum planes with an appropriate supplementary condition between deforming parameters. Moreover, there is a skew-symmetric non-degenerate bilinear map with which a one-parameter sub-family of linear connections are compatible. We shall also show that the skew-symmetric map resembles the symplectic 2-form of an ordinary manifold and induces a metric and the skew derivatives [27] of the h-deformed quantum plane. We shall compare the results of the h-deformed quantum plane with those of the q-deformed one. In section 4, we shall investigate the geometry of the extended h-deformed quantum plane. It turns out that the extended h-deformed quantum plane has a unique metric-compatible torsion-free linear connection; it is a non-commutative version of the Poincaré half-plane, a surface of constant negative Gaussian curvature. This can be shown explicitly by a change of generators.

2. Metric-compatible linear connections

2.1. Linear connections

Let $\mathcal{A}$ be an associative algebra with the identity 1. Let (Ω_u^*, d_u) be the universal differential calculus over $\mathcal{A}$. Then every other differential calculus over $\mathcal{A}$ can be obtained as a quotient of it. We suppose that there exists a bimodule of 1-forms Ω^1 and a map d of $\mathcal{A}$ into Ω^1. Then we can find an $\mathcal{A}$-bimodule homomorphism $\phi_1 : \Omega_u^1 \to \Omega^1$ such that $\phi_1 \circ d_u = d$. For integers $n \geqslant 2$, Ω^n is defined to be the quotient space

$$\Omega^n \equiv \frac{\Omega_u^n}{\langle d_u(\operatorname{Ker}\phi_{n-1})\rangle} \tag{2.1.1}$$

where ϕ_{n-1} is the projection map from Ω_u^{n-1} to Ω^{n-1} and $\langle d_u(\operatorname{Ker}\phi_{n-1})\rangle$ is the $\mathcal{A}$-bimodule generated by $d_u(\operatorname{Ker}\phi_{n-1})$. This construction can be summarized in the following commutative diagram

$$\begin{array}{ccccccc} \mathcal{A} & \xrightarrow{d_u} & \Omega_u^1 & \xrightarrow{d_u} & \Omega_u^2 & \xrightarrow{d_u} & \cdots \\ \| & & \phi_1\downarrow & & \phi_2\downarrow & & \\ \mathcal{A} & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{d} & \cdots. \end{array} \tag{2.1.2}$$

We shall be mainly interested here in the bimodules Ω^1 and Ω^2. Since

$$\Omega^2_u = \Omega^1_u \otimes_{\mathcal{A}} \Omega^1_u \tag{2.1.3}$$

there is an exact sequence of $\mathcal{A}$-bimodules

$$0 \to \mathcal{K} \hookrightarrow \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \xrightarrow{\pi} \Omega^2 \to 0 \tag{2.1.4}$$

where

$$\mathcal{K} = (\phi_1 \otimes \phi_1)(\langle d_u \mathrm{Ker}\,\phi_1 \rangle) = (\phi_1 \otimes \phi_1)(d_u \mathrm{Ker}\,\phi_1). \tag{2.1.5}$$

The definition of linear connection we use [9, 10, 18, 25] makes full use of the bimodule structure of Ω^1. It is defined to be a map

$$D : \Omega^1 \longrightarrow \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \tag{2.1.6}$$

satisfying the two Leibniz rules

$$D(f\xi) = df \otimes \xi + f D\xi \tag{2.1.7}$$

$$D(\xi f) = \sigma(\xi \otimes df) + (D\xi) f \tag{2.1.8}$$

where $f \in \mathcal{A}$, $\xi \in \Omega^1$ and σ is a map from $\Omega^1 \otimes_A \Omega^1$ to $\Omega^1 \otimes_A \Omega^1$ which generalizes the permutation. For the consistency of the definition of a linear connection, σ must be assumed to be $\mathcal{A}$-bilinear. That is, we must have, for $f \in A$ and $\xi, \eta \in \Omega^1$,

$$\sigma(f\xi \otimes \eta) = f\sigma(\xi \otimes \eta) \qquad \sigma(\xi \otimes \eta f) = \sigma(\xi \otimes \eta) f. \tag{2.1.9}$$

The map $\Theta : \Omega^1 \to \Omega^2$ defined by $\Theta = d - \pi \circ D$ is the torsion of the linear connection D. It is $\mathcal{A}$-bilinear only if σ is assumed to satisfy the condition [9]

$$\pi \circ (\sigma + 1) = 0. \tag{2.1.10}$$

A linear connection D can be extended to a linear map

$$D : \Omega^1 \otimes_A \Omega^1 \longrightarrow \Omega^1 \otimes_A \Omega^1 \otimes_A \Omega^1 \tag{2.1.11}$$

satisfying

$$D(\xi \otimes \eta) = D\xi \otimes \eta + \sigma_{12}(\xi \otimes D\eta) \tag{2.1.12}$$

for $\xi, \eta \in \Omega^1$, where $\sigma_{12} = \sigma \otimes 1$.

A $\mathcal{A}$-bilinear map

$$\Omega^1 \otimes_{\mathcal{A}} \Omega^1 \xrightarrow{g} \mathcal{A} \tag{2.1.13}$$

is called non-degenerate whenever $g(\xi \otimes \eta) = 0$ for all $\eta \in \Omega^1$ implies that $\xi = 0$ and $g(\xi \otimes \eta) = 0$ for all $\xi \in \Omega^1$ implies that $\eta = 0$. A metric is a non-degenerate $\mathcal{A}$-bilinear map. A metric is called symmetric (skew-symmetric) if $g \circ \sigma = g$ ($g \circ \sigma = -g$).

A linear connection D is said to be compatible with a metric g if the condition

$$(1 \otimes g) \circ D = d \circ g \tag{2.1.14}$$

is satisfied.

The curvature is defined to be the map

$$\pi_{12} D^2 : \Omega^1 \to \Omega^2 \otimes_{\mathcal{A}} \Omega^1 \tag{2.1.15}$$

where $\pi_{12} = \pi \otimes 1$. This map is left $\mathcal{A}$-linear if the torsion is free, but it is not, in general, right $\mathcal{A}$-linear [10]. There is at the moment no general consensus of the correct definition of the curvature respecting the bimodule structure of the linear connection, but since we are primarily interested in the first-order effects in the commutative limit, we can identify the curvature with the operator $\pi_{12} D^2$.

We define the Ricci map

$$\Omega^1 \xrightarrow{\text{Ric}} \Omega^1 \tag{2.1.16}$$

by $\text{Ric} = -(1 \otimes g)D^2$.

2.2. The Stehbein formalism

To initiate the construction in the previous subsection, we suppose that the algebra $\mathcal{A}$ is non-commutative and define the bimodule of 1-forms using a set of inner derivations [8]. For each positive integer n let λ_i be a set of n linearly independent elements of $\mathcal{A}$ and define the derivations by

$$e_i = \text{ad}\ \lambda_i. \tag{2.2.1}$$

For any $f \in \mathcal{A}$, we define the 1-form df by

$$df(e_i) = e_i f = [\lambda_i, f]. \tag{2.2.2}$$

The Ω^1 is then defined to be the $\mathcal{A}$-bimodule $\langle d\mathcal{A}\rangle$ generated by the image of d. Any element of Ω^1 is the sum of elements of the form $f\,dg$ or, equivalently using the Leibniz rule, of the form $(df)g$. We define

$$(f\,dg)(e_i) = f e_i g \qquad ((dg)f)(e_i) = (e_i g)f. \tag{2.2.3}$$

We suppose that there exists a set of n elements θ^i of Ω^1, called [8, 20] a 'frame' or 'Stehbein' as the non-commutative equivalent of a 'moving frame' or n-bein, such that

$$\theta^i(e_j) = \delta^i_j. \tag{2.2.4}$$

Then it follows easily that θ^i commute with the elements $f \in \mathcal{A}$,

$$f\theta^i = \theta^i f \tag{2.2.5}$$

and that Ω^1 is free of rank n as a left or right module. Hence, the exact sequence in equation (2.1.4) splits. Let $\jmath$ be the splitting map and write [20]

$$\jmath \circ \pi(\theta^i \otimes \theta^j) = P^{ij}{}_{kl}\theta^k \otimes \theta^l. \tag{2.2.6}$$

The coefficients $P^{ij}{}_{kl}$ depend on the map $\jmath$ and belong to the centre $\mathcal{Z}(\mathcal{A})$ of $\mathcal{A}$. Since π is a projection we have

$$P^{ij}{}_{mn}P^{mn}{}_{kl} = P^{ij}{}_{kl} \tag{2.2.7}$$

and the product $\theta^i\theta^j$ satisfies the condition

$$\theta^i\theta^j = P^{ij}{}_{kl}\theta^k\theta^l. \tag{2.2.8}$$

If we define $\theta = -\lambda_i\theta^i$, then it follows that

$$df = -[\theta, f] \tag{2.2.9}$$

and thus, as an $\mathcal{A}$-bimodule, the one element θ generates Ω^1.

If the θ^i exist, then it can be shown [8, 20] that λ_i must satisfy the equation

$$2\lambda_k\lambda_l P^{kl}{}_{ij} - \lambda_k F^k{}_{ij} - K_{ij} = 0 \tag{2.2.10}$$

with F^k_{ij} and K_{ij} complex numbers. Associated to this equation there is a modified Yang–Baxter equation [20]. The structure elements $C^i{}_{jk}$ are defined by the equation

$$\mathrm{d}\theta^i = -\tfrac{1}{2}C^i{}_{jk}\theta^j\theta^k. \tag{2.2.11}$$

They are related to the coefficients of equation (2.2.10) by the identity

$$C^i{}_{jk} = F^i{}_{jk} - 2\lambda_l P^{(li)}{}_{jk}. \tag{2.2.12}$$

Consistent with equation (2.2.8), we shall impose the conditions

$$P^{ij}{}_{kl} C^m{}_{ij} = C^m{}_{kl} \qquad P^{ij}{}_{kl} K_{ij} = K_{kl}. \tag{2.2.13}$$

Using the Stehbein, we now introduce a connection and a torsion 2-form

$$D\theta^i = -\omega^i{}_{jk}\theta^j \otimes \theta^k \tag{2.2.14}$$

$$\Theta^i = \mathrm{d}\theta^i - \pi \circ D\theta^i \tag{2.2.15}$$

as well as a metric

$$g(\theta^i \otimes \theta^j) = g^{ij}. \tag{2.2.16}$$

The coefficients $\omega^i{}_{jk}$ and g^{ij} must lie in $\mathcal{A}$. Since g is $\mathcal{A}$-bilinear and because of the condition (2.2.5), g^{ij} must lie in the centre $\mathcal{Z}(\mathcal{A})$. We also write

$$\sigma(\theta^i \otimes \theta^j) = S^{ij}{}_{kl}\theta^k \otimes \theta^l. \tag{2.2.17}$$

Then again, by the $\mathcal{A}$-bilinearity of σ, the coefficients $S^{ij}{}_{kl}$ also lie in $\mathcal{Z}(\mathcal{A})$. The condition (2.2.10) becomes

$$(S^{ij}{}_{kl} + \delta^i_k \delta^j_l) P^{kl}{}_{mn} = 0. \tag{2.2.18}$$

Using this notation, the metric-compatibility of a connection D is expressed as

$$\omega^i{}_{jk} + \omega_{kl}{}^m S^{il}{}_{jm} = 0. \tag{2.2.19}$$

The condition that the connection be torsion-free is given by

$$(\omega^i{}_{jk} - \tfrac{1}{2} C^i{}_{jk}) P^{jk}{}_{lm} = 0. \tag{2.2.20}$$

The curvature $\pi_{12} D^2$ can be written in terms of the frame as

$$\pi_{12} D^2 \theta^i = -\tfrac{1}{2} R^i{}_{jkl} \theta^k \theta^l \otimes \theta^j \tag{2.2.21}$$

and we have

$$\mathrm{Ric}(\theta^i) = \tfrac{1}{2} R^i{}_{jkl} \theta^k g(\theta^l \otimes \theta^j). \tag{2.2.22}$$

It is given by

$$\mathrm{Ric}(\theta^i) = R^i{}_j \theta^j. \tag{2.2.23}$$

3. The *h*-deformed quantum plane

3.1. Linear connections

The h-deformed quantum plane is an associative algebra $\mathcal{A}$ generated by non-commuting elements ('coordinates') x and y such that

$$xy - yx = hy^2 \tag{3.1.1}$$

where h is a deformation parameter. The quantum group $GL_h(2)$ is the symmetry group of the h-deformed plane as is $GL_q(2)$ for the q-deformed quantum plane [7, 16]. Let

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_h(2).$$

The commutation relations between the matrix elements of the quantum group are given by

$$\begin{aligned} AB - BA &= h\delta - hA^2 \\ AC - CA &= hC^2 \\ AD - DA &= hCD - hCA \\ BC - CB &= hCD + hAC \\ BD - DB &= hD^2 - h\delta \\ CD - DC &= -hC^2 \end{aligned}$$

where the quantum determinant

$$\delta = AD - CB - hCD = DA - CB - hCA \tag{3.1.2}$$

is central.

The R-matrix associated with this quantum group and which solves the quantum Yang–Baxter equation

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} \tag{3.1.3}$$

is given by

$$\hat{R} = \begin{pmatrix} 1 & -h & h & h^2 \\ 0 & 0 & 1 & h \\ 0 & 1 & 0 & -h \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.1.4}$$

The covariant differential calculus on the quantum plane can be found [4] by the method of Wess and Zumino [27]. The results to be used in this work can be summarized as follows. For $x^i = (x, y)$ and $\xi^i = dx^i = (\xi, \eta)$ we have

$$\begin{aligned} x^a x^b &= \hat{R}^{ab}{}_{cd} x^c x^d \qquad & x^a \xi^b &= \hat{R}^{ab}{}_{cd} \xi^c x^d \\ \xi^a \xi^b &= -\hat{R}^{ab}{}_{cd} \xi^c \xi^d \qquad & \partial_a x^b &= \delta^b_a + \hat{R}^{bd}{}_{ac} x^c \partial_d. \end{aligned} \tag{3.1.5}$$

The second and third equations are written explicitly as

$$\begin{aligned} x\xi &= \xi x - h\xi y + h\eta x + h^2 \eta y \qquad & x\eta &= \eta x + h\eta y \\ y\xi &= \xi y - h\eta y \qquad & y\eta &= \eta y \end{aligned} \tag{3.1.6}$$

and

$$\xi^2 = h\xi\eta \qquad \xi\eta = -\eta\xi \qquad \eta^2 = 0. \tag{3.1.7}$$

Now, as in the q-deformed quantum plane [9], the action of a linear connection on the second equation of the above relations generically results in the following relations:

$$\xi^a \otimes \xi^b = \hat{R}^{ab}{}_{cd} \sigma(\xi^c \otimes \xi^d) \qquad x^a D\xi^b = \hat{R}^{ab}{}_{cd} (D\xi^c) x^d. \tag{3.1.8}$$

Since $\hat{R}^{-1} = \hat{R}$, the first equation is verified when σ transforms as $\hat{R}$, i.e.

$$\begin{aligned} \sigma(\xi \otimes \xi) &= \xi \otimes \xi - h\xi \otimes \eta + h\eta \otimes \xi + h^2 \eta \otimes \eta \\ \sigma(\xi \otimes \eta) &= \eta \otimes \xi + h\eta \otimes \eta \\ \sigma(\eta \otimes \xi) &= \xi \otimes \eta - h\eta \otimes \eta \\ \sigma(\eta \otimes \eta) &= \eta \otimes \eta. \end{aligned} \tag{3.1.9}$$

The definition of the map σ now is extended to the whole space $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$ by the $\mathcal{A}$-linearity. Then it is easy to see that σ satisfies equation (2.1.10) and

$$\sigma^2 = 1. \tag{3.1.10}$$

Also from the quantum Yang–Baxter equation of $\hat{R}$ in equation (3.1.3)) it follows that

$$\sigma_{12}\sigma_{23}\sigma_{12} = \sigma_{23}\sigma_{12}\sigma_{23}. \tag{3.1.11}$$

We are now in a position to exhibit the general linear connection D on Ω^1, as in the q-deformed quantum plane. We introduce the 1-form

$$\kappa = x\eta - y\xi - hy\eta \tag{3.1.12}$$

which is covariant under the action of $SL_h(2)$. Then we have

$$\begin{aligned} &x\kappa = \kappa x \qquad y\kappa = \kappa y \\ &\xi\kappa = -\kappa\xi \qquad \eta\kappa = -\kappa\eta \end{aligned} \tag{3.1.13}$$

and

$$\kappa^2 = 0. \tag{3.1.14}$$

κ also obeys

$$\begin{aligned} &\sigma(\xi\otimes\kappa) = \kappa\otimes\xi \qquad \sigma(\kappa\otimes\xi) = \xi\otimes\kappa \\ &\sigma(\eta\otimes\kappa) = \kappa\otimes\eta \qquad \sigma(\kappa\otimes\eta) = \eta\otimes\kappa \\ &\sigma(\kappa\otimes\kappa) = \kappa\otimes\kappa. \end{aligned} \tag{3.1.15}$$

A solution $D\xi^b$ to the second equation in (3.1.8) can be immediately read off from the second equation of (3.1.5)

$$D\xi^a = \rho(\xi^a\otimes\kappa + \kappa\otimes\xi^a) \tag{3.1.16}$$

where ρ is a real parameter. Also, the first equation in (3.1.5) suggests a solution of the form

$$D\xi^a = \mu x^a\varpi \tag{3.1.17}$$

where μ is a real parameter and ϖ is any element in $\Omega^1\otimes_{\mathcal{A}}\Omega^1$ such that $\pi\varpi = 0$ and

$$x^a\varpi = \varpi x^a. \tag{3.1.18}$$

From equation (3.1.13), it is easy to see that $\varpi = \kappa\otimes\kappa$. Then the general torsion-free linear connection D is given by

$$D\xi^a = \mu x^a\kappa\otimes\kappa + \rho(\xi^a\otimes\kappa + \kappa\otimes\xi^a). \tag{3.1.19}$$

This two-parameter solution has been also found by Khorrami *et al* [15]. Now it is natural to investigate other possible ϖ in various cases. For this we extend the h-deformed quantum plane to the two-parameter case [4] on which the two-parameter quantum group $GL_{h,h'}$ acts. In this case, we have the same equations as in (3.1.5) and (3.1.8) with $\hat{R}$ replaced by

$$\hat{R} = \begin{pmatrix} 1 & -h' & h' & hh' \\ 0 & 0 & 1 & h \\ 0 & 1 & 0 & -h \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.1.20}$$

A straightforward calculation yields ϖ as follows. For $h' = nh$ $(n = 2, 3, 4, \ldots)$

$$\varpi = y^{n-2}(\kappa\otimes\eta + \eta\otimes\kappa) \tag{3.1.21}$$

and for $h' = \frac{1}{2}nh$ $(n = 2, 3, 4, \ldots)$

$$\varpi = y^{n-2}\kappa\otimes\kappa - \frac{n-2}{2}hy^{n-1}\eta\otimes\kappa. \tag{3.1.22}$$

Then we have a two-parameter family of torsion-free linear connections for $h' = nh$ $(n = 2, 3, 4, \ldots)$

$$D\xi^a = \mu x^a (y^{2n-2}\kappa \otimes \kappa - (n-1)hy^{2n-1}\eta \otimes \kappa) + \rho x^a y^{n-2}(\kappa \otimes \eta + \eta \otimes \kappa) \tag{3.1.23}$$

and a one-parameter family of torsion-free linear connections for $h' = \frac{1}{2}nh$ $(n = 3, 5, 7, \ldots)$

$$D\xi^a = \mu x^a \left(y^{n-2}\kappa \otimes \kappa - \frac{n-2}{2} h y^{n-1}\eta \otimes \kappa \right). \tag{3.1.24}$$

The supplementary condition $h' = nh$ or $h' = \frac{1}{2}nh$ corresponds to $p = q^n$ for the case [14] of the q-deformed quantum plane with the two-parameter quantum group $GL_{q,p}(2)$ where the linear connection is given by $D\xi^a = \mu x^a x^{n-1} y^{n-1} \kappa \otimes \kappa$.

In the next subsection, we consider the first term of the two-parameter family of torsion-free linear connections in equation (3.1.19)

$$D\xi^a = \mu x^a \kappa \otimes \kappa \tag{3.1.25}$$

since it is compatible with a skew-symmetric non-degenerate bilinear map. From the linear connection we have the curvature

$$\pi_{12} D^2 \xi^a = -\Omega^a{}_b \otimes \xi^b \tag{3.1.26}$$

where the 2-form $\Omega^a{}_b$ is given by

$$\Omega^a{}_b = 4\mu \begin{pmatrix} xy & -x^2 + hxy \\ y^2 & -yx + hy^2 \end{pmatrix} \xi\eta. \tag{3.1.27}$$

The 1-form κ satisfies the equation $D^2\kappa = 0$.

3.2. The symplectic 2-form

In this subsection, we shall use the expression 'symplectic 2-form' and 'skew-symmetric metric' synonymously and denote it by Λ since a skew-symmetric metric on the h-deformed quantum plane resembles a symplectic 2-form as in the ordinary geometry. A symmetric metric will be denoted simply as a metric as in ordinary geometry.

It has been shown that no metric can exist in the case of the q-deformed quantum plane [9]. However, the h-deformed quantum plane has a better geometrical structure than the q-deformed quantum plane and it does have a metric. In fact, the h-deformed quantum plane has a symplectic 2-form, with which a metric can be associated. The symplectic 2-form of the h-deformed quantum plane is given in a matrix form as

$$\Lambda(\xi^a \otimes \xi^b) \equiv \Lambda^{ab} = \begin{pmatrix} h & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.2.1}$$

Now it is straightforward to show that in the particular case when the covariant derivative is given by $D\xi^a = \mu x^a \kappa \otimes \kappa$ we have

$$(1 \otimes \Lambda) D(\xi^a \otimes \xi^b) = d\Lambda^{ab} = 0 \tag{3.2.2}$$

and for $\sigma_{23} = 1 \otimes \sigma$

$$\begin{aligned} &(1 \otimes \Lambda)\sigma_{12}\sigma_{23}(\xi \otimes \xi \otimes \xi) = h\xi \\ &(1 \otimes \Lambda)\sigma_{12}\sigma_{23}(\xi \otimes \xi \otimes \eta) = h\eta \\ &(1 \otimes \Lambda)\sigma_{12}\sigma_{23}(\xi \otimes \eta \otimes \xi) = \xi \\ &(1 \otimes \Lambda)\sigma_{12}\sigma_{23}(\xi \otimes \eta \otimes \eta) = \eta \\ &(1 \otimes \Lambda)\sigma_{12}\sigma_{23}(\eta \otimes \xi \otimes \xi) = -\xi \end{aligned} \tag{3.2.3}$$

$$(1 \otimes \Lambda)\sigma_{12}\sigma_{23}(\eta \otimes \xi \otimes \eta) = -\eta$$
$$(1 \otimes \Lambda)\sigma_{12}\sigma_{23}(\eta \otimes \eta \otimes \xi) = 0$$
$$(1 \otimes \Lambda)\sigma_{12}\sigma_{23}(\eta \otimes \eta \otimes \eta) = 0.$$

From these relations, it follows that the symplectic 2-form Λ satisfies the compatibility condition in equation (2.1.14), while the symplectic 2-form $\Lambda = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}$ of the q-deformed quantum plane does not.

We define the h-deformed symplectic group by

$$Sp_h(1) = \{T \in GL_h(2) \mid T\Lambda T^t = \Lambda\}. \tag{3.2.4}$$

Equivalently, the symplectic 2-form Λ is preserved under the action of $Sp_h(1)$, i.e.

$$\Lambda(\xi'^a \otimes \xi'^b) = \Lambda(\xi^a \otimes \xi^b) \tag{3.2.5}$$

under the transformation

$$\begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

From equation (3.1.5) it follows that $Sp_h(1) = SL_h(2)$, which is consistent with the commutative limit when $h \to 0$.

In ordinary symplectic geometry, metrics can be defined by a symplectic 2-form together with a complex structure and these all together result in a Hermitian inner product. One can do the same in the h-deformed quantum plane. We can define an $\mathcal{A}$-linear map $J : \Omega^1 \to \Omega^1$ by

$$J\xi = \mathrm{i}\xi \qquad J\eta = -\mathrm{i}\eta \tag{3.2.6}$$

where $\mathrm{i} = \sqrt{-1}$. The map J satisfies $J^2 = -1$ and can be regarded as the complex structure of the h-deformed quantum plane. Associated with the symplectic 2-form Λ, there is a metric g satisfying the following relation, for $\xi, \eta \in \Omega^1$,

$$g(J\xi \otimes \eta) = \Lambda(\xi \otimes \eta) \tag{3.2.7}$$

which can be written in matrix form as

$$g = \begin{pmatrix} -\mathrm{i}h & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}. \tag{3.2.8}$$

On the other hand, there is another metric g' defined by

$$g'(\xi \otimes \eta) = \Lambda(\xi \otimes J\eta) \tag{3.2.9}$$

which is, in matrix form,

$$g' = \begin{pmatrix} \mathrm{i}h & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}. \tag{3.2.10}$$

The two metrics are related by the condition

$$g(J\xi \otimes J\eta) = g'(\xi \otimes \eta) \tag{3.2.11}$$

for any $\xi, \eta \in \Omega^1$ and agree when $h \to 0$. These metrics, however, are not compatible with the linear connection D. In fact there is no metric on the h-deformed plane with respect to which D is compatible. Such a metric can be found, however, if we extend the h-deformed quantum plane as in the next section. In order to compare them with the commutative-limit case let us define

$$\vartheta^1 = \frac{1}{\sqrt{2}}(\xi + \mathrm{i}\eta) \qquad \vartheta^2 = \frac{\mathrm{i}}{\sqrt{2}}(\xi - \mathrm{i}\eta). \tag{3.2.12}$$

Then it is easy to see that

$$J\vartheta^1 = \vartheta^2 \qquad J\vartheta^2 = -\vartheta^1. \tag{3.2.13}$$

With respect to $\{\vartheta^1, \vartheta^2\}$, the two metrics g, g', and the symplectic 2-form Λ can be expressed as follows

$$g = \begin{pmatrix} 1 - \frac{1}{2}\mathrm{i}h & \frac{1}{2}h \\ \frac{1}{2}h & 1 + \frac{1}{2}\mathrm{i}h \end{pmatrix}, \; g' = \begin{pmatrix} 1 + \frac{1}{2}\mathrm{i}h & -\frac{1}{2}h \\ -\frac{1}{2}h & 1 - \frac{1}{2}\mathrm{i}h \end{pmatrix}, \; \Lambda = \begin{pmatrix} \frac{1}{2}h & 1 + \frac{1}{2}\mathrm{i}h \\ -1 + \frac{1}{2}\mathrm{i}h & -\frac{1}{2}h \end{pmatrix}. \tag{3.2.14}$$

If we define

$$H = g' + \mathrm{i}\Lambda \tag{3.2.15}$$

the map H goes over to the usual Hermitian inner product on the complex 2-plane $\mathbb{C}^2$ in the commutative limit. Moreover, it is interesting to see that if we let $\eta_a = (-\eta, \xi + h\eta)$ and define the skew derivative ∂_a by

$$\partial_a f = \Lambda(\eta_a \otimes df) \tag{3.2.16}$$

then ∂_a satisfy the second equation of (3.1.5) given by Wess and Zumino [27]. Thus the skew derivatives ∂_a arise as Hamiltonian vector fields in the h-deformed quantum plane. This is not the case for the q-deformed quantum plane. In fact, if $x^a\xi^b = \hat{R}^{ab}{}_{cd}\xi^c x^d$, there should be elements $\eta_a \in \Omega^1$ such that

$$\eta_a x^b = \hat{R}^{bc}{}_{ad}x^d\eta_c \tag{3.2.17}$$

for the skew derivatives to be induced from the symplectic 2-form Λ as above. However, there are no such η_a in the case of the q-deformed quantum plane.

4. The extended *h*-deformed plane

4.1. Linear connections

The extended h-deformed quantum plane is an associative algebra $\mathcal{A}$ generated by x, y, x^{-1}, y^{-1} satisfying equation (3.1.1). The extended h-deformed plane is also more interesting than the extended q-deformed one from the point of view of geometry since the metric and linear connection it supports have an interesting commutative limit.

If $\mathcal{A}$ is a unital $*$-algebra and x and y are Hermitian elements, then $h \in \mathrm{i}\mathbb{R}$. Equation (3.1.1) can be written as $[x, y^{-1}] = -h$. In this form we see that the algebra has the structure of the Heisenberg algebra with the parameter h playing the role of $\hbar$ but the differential calculus (3.1.6) is not 'natural' from this point of view. If we introduce

$$u = xy^{-1} + \tfrac{1}{2}h \qquad v = y^{-2} \tag{4.1.1}$$

then the commutation relation becomes

$$[u, v] = -2hv. \tag{4.1.2}$$

This choice of generators is useful in studying the commutative limit. If x and y are Hermitian, then so are u and v.

We can write (4.1.2) also as $[u, (1/2)\log v] = -h$ if we introduce the formal element $\log v$. We see then that

$$x' = u \qquad y' = \frac{2}{\log v}$$

also satisfy the commutation relations (3.1.1). The algebra cannot then be uniquely defined by the commutation relations. In fact, von Neumann proved that only by using additional topological conditions could one deduce the uniqueness of the representation of the Heisenberg commutation relations.

A (real) frame can be written in terms of the generators in equation (3.1.1) as

$$\theta^1 = y\xi - (x - hy)\eta \qquad \theta^2 = 2y^{-1}\eta \tag{4.1.3}$$

and in terms of the generators in equation (4.1.2) as

$$\theta^1 = v^{-1}\,\mathrm{d}u \qquad \theta^2 = -v^{-1}\,\mathrm{d}v. \tag{4.1.4}$$

The original basis (ξ, η) can be written in terms of the θ^a as

$$2\xi = 2y^{-1}\theta^1 + x\theta^2 \qquad 2\eta = y\theta^2. \tag{4.1.5}$$

θ^a satisfy the commutation relations (2.2.5) as well as the relations

$$(\theta^1)^2 = 0 \qquad (\theta^2)^2 = 0 \qquad \theta^1\theta^2 + \theta^2\theta^1 = 0. \tag{4.1.6}$$

From equation (2.2.6) we see then that $P^{ab}{}_{cd}$ is given by

$$P^{ab}{}_{cd} = \tfrac{1}{2}(\delta^a_c\delta^b_d - \delta^b_c\delta^a_d) \tag{4.1.7}$$

and, therefore, from equation (2.2.12), we have $C^a{}_{bc} = F^a{}_{bc}$. In particular, $C^a{}_{bc}$ are real numbers. The differentials $\mathrm{d}\theta^a$ are given by equation (2.2.11) with

$$C^1{}_{12} = -C^1{}_{21} = 1 \qquad C^2{}_{ab} = 0. \tag{4.1.8}$$

If we introduce the derivations $e_a = \mathrm{ad}\,\lambda_a$ with

$$\lambda_1 = \frac{1}{2h}y^{-2} = \frac{1}{2h}v \qquad \lambda_2 = \frac{1}{2h}xy^{-1} + \frac{1}{4} = \frac{1}{2h}u \tag{4.1.9}$$

we see that equation (2.2.4) is satisfied. We can conclude from equation (2.2.10) that λ_a must form a (real) Lie algebra. We then have from equation (4.1.2)

$$[\lambda_1, \lambda_2] = \lambda_1. \tag{4.1.10}$$

The λ_a form a solvable Lie algebra.

The 'Dirac operator' [6] in equation (2.2.9) is given by

$$\theta = -\frac{1}{2h}y^{-1}\xi - \frac{1}{2h}(x - hy)y^{-2}\eta = -\frac{1}{2h}(\mathrm{d}u - uv^{-1}\,\mathrm{d}v). \tag{4.1.11}$$

A straightforward calculation yields $\mathrm{d}\theta + \theta^2 = 0$.

We introduce a metric and we set $g(\theta^a \otimes \theta^b) = g^{ab}$. From the bilinearity condition on g and the relations (3.1.1) we see that g^{ab} must be complex numbers. If we wish the metric to be real then g^{ab} must be real numbers. By a trivial change of basis we can suppose that $g^{ab} = \delta^{ab}$. We then have in terms of the generators x and y

$$\begin{aligned} &g(\xi \otimes \xi) = y^{-2} + x^2/4 \qquad g(\xi \otimes \eta) = xy/4 \\ &g(\eta \otimes \xi) = yx/4 \qquad g(\eta \otimes \eta) = y^2/4 \end{aligned} \tag{4.1.12}$$

and in terms of the generators u and v

$$\begin{aligned} &g(\mathrm{d}u \otimes \mathrm{d}u) = v^2 \qquad g(\mathrm{d}u \otimes \mathrm{d}v) = 0 \\ &g(\mathrm{d}v \otimes \mathrm{d}u) = 0 \qquad g(\mathrm{d}v \otimes \mathrm{d}v) = v^2. \end{aligned} \tag{4.1.13}$$

A flat metric-compatible linear connection is given by

$$D\theta^a = 0. \tag{4.1.14}$$

It has torsion, given by

$$\Theta^1 = -\theta^1\theta^2 \qquad \Theta^2 = 0. \tag{4.1.15}$$

For the generic solution σ in equation (3.1.9), the unique torsion-free, metric-compatible linear connection is given by

$$D\theta^1 = -\theta^1 \otimes \theta^2 \qquad D\theta^2 = \theta^1 \otimes \theta^1. \tag{4.1.16}$$

This D is also compatible with the symplectic 2-form Λ given in equation (3.2.1): $D\Lambda = 0$.

The curvature map defined by equation (2.2.21) becomes

$$\pi_{12}D^2\theta^1 = \theta^1\theta^2 \otimes \theta^2 \qquad \pi_{12}D^2\theta^2 = -\theta^1\theta^2 \otimes \theta^1. \tag{4.1.17}$$

If one sets as usual $R_{abcd} = g_{ae}R^e{}_{bcd}$ then one finds that the Gaussian curvature is given by

$$R_{1212} = -1. \tag{4.1.18}$$

The coefficients R_{abcd} satisfy the usual symmetries of the coefficients of a Riemann curvature tensor. The coefficients of the Ricci map are given by

$$R^a{}_b = \delta^a_b. \tag{4.1.19}$$

We choose now $n = 3$. Then it is of interest to introduce a third (Hermitian) element

$$w = -\tfrac{1}{2}(u^2 - 2hu + 1 + 2h^2)v^{-1} \tag{4.1.20}$$

of the algebra $\mathcal{A}$ and define

$$\lambda_3 = \frac{1}{2h}w. \tag{4.1.21}$$

The $\lambda_i = (\lambda_a, \lambda_3)$ still form a Lie algebra

$$[\lambda_1, \lambda_2] = \lambda_1 \qquad [\lambda_2, \lambda_3] = \lambda_3 \qquad [\lambda_3, \lambda_1] = \lambda_2. \tag{4.1.22}$$

A straightforward calculation yields

$$\begin{aligned} &e_1u = v \qquad e_1v = 0 \qquad e_1w = -u \\ &e_2u = 0 \qquad e_2v = -v \qquad e_2w = w \\ &e_3u = -w \qquad e_3v = u \qquad e_3w = 0. \end{aligned} \tag{4.1.23}$$

e_i satisfy the same commutation relations

$$[e_1, e_2] = e_1 \qquad [e_2, e_3] = e_3 \qquad [e_3, e_1] = e_2 \tag{4.1.24}$$

as λ_i. They are real in the sense that the derivation $e_i f$ of an Hermitian element f is Hermitian.

The Lie algebra in equation (4.1.24) is a real form of $SL(2, \mathbb{C})$, different from the Lie algebra of SO_3. We have found a frame with two generators since the Poincaré half-plane is a parallelizable manifold and the module of 1-forms is a free (left or right) module. This is not so in the case of the 2-sphere [18]; the module of 1-forms in this case is a non-trivial submodule of a free module of rank three. The Lie algebra of Killing vector fields of the Poincaré half-plane and the sphere are different real realizations of $SL(2, \mathbb{C})$.

A differential calculus can be defined using the three 1-forms θ^i dual to the derivations e_i. An analogous situation was discussed in the case of the q-deformed plane [8]. From equation (4.1.23) we conclude that

$$\mathrm{d}u = v\theta^1 - w\theta^3 \qquad \mathrm{d}v = -v\theta^2 + u\theta^3 \tag{4.1.25}$$

to which we can add

$$\mathrm{d}w = -u\theta^1 + w\theta^2. \tag{4.1.26}$$

The second of equations (4.1.25) is a trivial consequence of the commutation relations, obtained by equating the differential of both sides of equation (4.1.2). The previous differential calculus with two generators is obtained formally by setting $\theta^3 = 0$ in equation (4.1.25). The commutation relations which define the module structure of Ω^1 are obtained from equation (4.1.25):

$$\begin{aligned} u\,\mathrm{d}u - \mathrm{d}u\,u &= -2h\,\mathrm{d}u - 4hw\theta^3 \qquad & v\,\mathrm{d}u - \mathrm{d}u\,v &= 2hu\theta^3 \\ u\,\mathrm{d}v - \mathrm{d}v\,u &= -2h\,\mathrm{d}v + 2hu\theta^3 \qquad & v\,\mathrm{d}v - \mathrm{d}v\,v &= 2hv\theta^3. \end{aligned} \tag{4.1.27}$$

Apart from the trivial relation which follows from the commutation relations these equations contain two cubic relations

$$\begin{aligned} &uv\,\mathrm{d}v - u\,\mathrm{d}v\,v = v\,\mathrm{d}u\,v - \mathrm{d}u\,v^2 \\ &vu\,\mathrm{d}u - v\,\mathrm{d}u\,u + 2hv\,\mathrm{d}u + 2v\,\mathrm{d}v\,w - 2\,\mathrm{d}v\,vw = 0. \end{aligned} \tag{4.1.28}$$

Provided that $h \neq 0$ the system equations (4.1.25) and (4.1.26) can be inverted to obtain equations for the θ^i in terms of $\mathrm{d}u$, $\mathrm{d}v$ and $\mathrm{d}w$:

$$\theta^1 = \frac{1}{2h}u^{-1}[w,\,\mathrm{d}u] \qquad \theta^2 = \frac{1}{2h}v^{-1}[u,\,\mathrm{d}v] \qquad \theta^3 = \frac{1}{2h}u^{-1}[v,\,\mathrm{d}u]. \tag{4.1.29}$$

This differential calculus has fewer relations than the one defined above. It lies between that defined by relations (4.1.6) and the universal differential calculus, which has a free algebra of forms with no relations.

If one defines the Lie derivative of an element $f \in \mathcal{A}$ as $L_X f = Xf$ then L_X can be extended to a derivation of the tensor algebra over Ω^1 just as in ordinary geometry. In particular, if ξ, η are 1-forms and i_X is the interior product then the Lie derivative $L_X\xi$ of ξ is given by

$$L_X\xi = \mathrm{d}i_X\xi + i_X\,\mathrm{d}\xi \tag{4.1.30}$$

and $L_X(\xi f \otimes \eta) = L_X(\xi \otimes f\eta)$ is well defined.

A Killing derivation [18] is a derivation X such that the Lie derivative L_X of the metric g vanishes: $L_X g = 0$. Denote L_i as the Lie derivative with respect to the derivation e_i. Then it is easy to see that

$$\begin{aligned} L_1\theta^1 &= -\theta^2 \qquad & L_1\theta^2 &= 0 \\ L_2\theta^1 &= +\theta^1 \qquad & L_2\theta^2 &= 0 \\ L_3\theta^1 &= -v^{-1}w\theta^2 \qquad & L_3\theta^2 &= -v^{-1}u\theta^2 - \theta^1. \end{aligned} \tag{4.1.31}$$

From these formulae one can calculate the Lie derivative of the metric for the 'covariant form' $g = g_{ab}\theta^a \otimes \theta^b$:

$$\begin{aligned} L_1 g &= -(\theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1) \\ L_2 g &= 2\theta^1 \otimes \theta^1 \\ L_3 g &= -(1 + v^{-1}w)(\theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1) - 2v^{-1}u\theta^2 \otimes \theta^2. \end{aligned} \tag{4.1.32}$$

That is, none of the derivations e_i is a Killing derivation.

4.2. The commutative limit of the extended plane

It is interesting to study the structure of the extended h-deformed quantum plane in the commutative limit. In terms of the commutative limits $\tilde{u}$, $\tilde{v}$ of the generators u, v of the

algebra $\mathcal{A}$ and the corresponding commutative limit $\tilde{\theta}^a$ of the frame, the metric is given by the line element

$$\mathrm{d}s^2 = (\tilde{\theta}^1)^2 + (\tilde{\theta}^2)^2 = \tilde{v}^{-2}(\mathrm{d}\tilde{u}^2 + \mathrm{d}\tilde{v}^2). \tag{4.2.1}$$

This is the metric of the Poincaré half-plane. The algebra $\mathcal{A}$ with the differential calculus defined by the relations (4.1.6) can be considered then as a non-commutative deformation of the Poincaré half-plane.

The derivations e_i define, in the commutative limit, three vector fields

$$X_i = \lim_{h\to 0} e_i. \tag{4.2.2}$$

If we define $\tilde{w}$ to be the commutative limit of w then

$$X_1 = \tilde{v}\partial_{\tilde{u}} \qquad X_2 = -\tilde{v}\partial_{\tilde{v}} \qquad X_3 = -\tilde{w}\partial_{\tilde{u}} + \tilde{u}\partial_{\tilde{v}}. \tag{4.2.3}$$

By construction these vector fields form a Lie algebra with the same commutation relations as the e_i. By equation (4.1.32) of the previous section we see that the X_i cannot be Killing vector fields. There is in fact no reason for them to be so. The Poincaré half-plane has, however, three Killing vector fields X'_i, given by

$$X'_1 = \partial_{\tilde{u}} \qquad X'_2 = \tilde{u}\partial_{\tilde{u}} + \tilde{v}\partial_{\tilde{v}} \qquad X'_3 = \tfrac{1}{2}(\tilde{v}^2 - \tilde{u}^2)\partial_{\tilde{u}} - \tilde{u}\tilde{v}\partial_{\tilde{v}}. \tag{4.2.4}$$

Define a map ϕ of the Poincaré half-plane into itself by

$$\phi(\tilde{u}) = \tilde{u}' = \tilde{u}\tilde{v}^{-1} \qquad \phi(\tilde{v}) = \tilde{v}' = \tilde{v}^{-1}. \tag{4.2.5}$$

This is a regular diffeomorphism. Indeed

$$\phi^2 = \phi \circ \phi = 1. \tag{4.2.6}$$

In the spirit of non-commutative geometry we consider $\tilde{u}$ and $\tilde{v}$ as generators of the algebra of functions on the Poincaré half-plane. In ordinary differential geometry a map ϕ of the manifold induces a map ϕ^* of the algebra of differential forms and a map ϕ_* of the vector fields. Since we shall not have occasion to refer to the manifold as such, we use the notation ϕ to designate the restriction of ϕ^* to the algebra of functions. Since we have

$$\phi_*\partial_{\tilde{u}} = \partial_{\tilde{u}'} = \tilde{v}\partial_{\tilde{u}} \qquad \phi_*\partial_{\tilde{v}} = \partial_{\tilde{v}'} = -\tilde{u}\tilde{v}\partial_{\tilde{u}} - \tilde{v}^2\partial_{\tilde{v}} \tag{4.2.7}$$

it is easy to see that

$$\phi_* X_i = X'_i. \tag{4.2.8}$$

The commutative limit of the derivations which defined the differential calculus are then related to the Killing vector fields in a simple way. We have not succeeded in constructing derivations of the algebra whose commutative limits are the Killing vector fields X'_i. The limit $h \to 0$ is a rather singular limit and it need not be true that an arbitrary vector field on the Poincaré half-plane is the limit of a derivation. The action of ϕ^* on the frame is given by

$$\phi^*\tilde{\theta}^1 = \tilde{v}'^{-1}\,\mathrm{d}\tilde{u}' = \tilde{v}\tilde{\theta}^1 + \tilde{u}\tilde{\theta}^2 \qquad \phi^*\tilde{\theta}^2 = -\tilde{v}'^{-1}\,\mathrm{d}\tilde{v}' = -\tilde{\theta}^2. \tag{4.2.9}$$

The vector fields X_i are Killing with respect to the metric

$$\mathrm{d}s^2 = (\phi^*\tilde{\theta}^1)^2 + (\phi^*\tilde{\theta}^2)^2. \tag{4.2.10}$$

The map ϕ can also be considered as a change of coordinates. In this case equations (4.2.1) and (4.2.10) describe the same line element in different coordinates systems. The components X_i^a of the vector fields X_i coincide with the components of the Killing vector fields in the new coordinate system and the components $X_i'^a$ of the vector fields X'_i coincide with the components of the Killing vector fields in the old coordinate system. We

have not really understood the role of the map ϕ nor why it appears. We have constructed the algebra $\mathcal{A}$ using generators and relations. This is the non-commutative version of the method of defining a curved manifold as an embedding in a higher-dimensional flat Euclidean space. It is known [13, 24] that the Poincaré half-plane cannot even be immersed in $\mathbb{R}^3$. This fact might somehow also be connected with the existence of the map ϕ.

The commutation relations (4.1.2) define on the Poincaré half-plane a Poisson structure

$$\{\tilde{u}, \tilde{v}\} = -2\tilde{v}. \tag{4.2.11}$$

Since the map ϕ is not a symplectomorphism it cannot be 'lifted' to a morphism of the algebra $\mathcal{A}$. There should be a relation [19] between the Poisson structure and the Riemann curvature. It is not evident from the present example, however, what this relation could be. The Poincaré half-plane has been used as an example of a classical and quantum phase space and as such has many interesting properties. For a discussion of this and reference to the previous literature we refer the reader to Emch *et al* [11]. The relation between the algebra of a free quantum particle on the Poincaré half-plane and the h-deformed algebra we have used has yet to be investigated.

5. Conclusion

The h-deformed quantum plane seems to have more geometrical structures than the q-deformed one. In the h-deformed quantum plane, there is a two-parameter family of torsion-free linear connections. The existence of a two-parameter family of torsion-free linear connections is quite general even within the set of two-parameter h-deformed quantum planes. Moreover, there is a skew-symmetric non-degenerate bilinear map with which a one-parameter sub-family of linear connections are compatible. The skew-symmetric map plays an important role. It resembles the symplectic 2-form and makes the linear connections symplectic. Moreover, it is interesting that the skew-symmetric map induces skew derivatives in the h-deformed quantum plane. We can also define a complex structure on the h-deformed quantum plane and construct a metric using this structure together with the skew-symmetric map as in the ordinary symplectic geometry [3]. However, it should be stressed that the metric is not compatible with the linear connections. A similar construction is not possible in the case of the q-deformed quantum plane.

The geometry of the Poincaré half-plane can be completely globally defined by the action of the $SL(2, \mathbb{R})$ group whose Lie algebra is given by the Killing vectors. From this point of view a complete classification of all Poisson structures on the Poincaré (Lobachevsky) half-plane as well as their possible 'quantum' deformations has been given by Leitenberger [12, 17]. We have analysed in detail the extended h-deformed quantum plane as a non-commutative version of the Poincaré half-plane; the roles of the derivations and the Stehbein are explicitly investigated.

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